Crystalline conjecture via K-theory

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1 Motivation and statement of the crystalline conjecture

1.1 Crystalline conjecture

TO DO: clarify where we're assuming that V is absolute unramified. Occasionally we need $V = W$, e.g. when we refer to the Hodge filtration on crystalline cohomology (it's really on de Rham cohomology, but they agree in this case), and when applying the Künneth formula in crystalline cohomology.

Let K be a complete discretely valued field of mixed characteristic $(0, p)$, with ring of integers V and perfect residue field k. Let X be a smooth projective V-scheme. Fontaine's crystalline conjecture states that there is a natural isomorphism between the p-adic étale cohomology of the generic fiber of X and the crystalline cohomology of the special fiber, after tensoring both up to the p-adic period ring B_{cris}:

$$\alpha : H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \simeq H^*_{\text{cris}}(X_k/W(k)) \otimes_{W(k)} B_{\text{cris}} \tag{1}$$

compatible with filtration, Galois action, and Frobenius. (Étale cohomology has a Galois action but trivial Frobenius and filtration; crystalline has a Frobenius and Hodge filtration but no Galois action.) It follows from this that the étale and crystalline cohomology of X, equipped with all available structure, completely determine each other: roughly, to get from one to the other, tensor up to B_{cris} and take invariants under all extra structure that you don't want to have.

The crystalline conjecture also has a more subtle integral version, stating that there is a natural almost isomorphism

$$\alpha : H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\text{cris}} \simeq H^*_{\text{cris}}(X/W(k)) \otimes_{W(k)} A_{\text{cris}}, \tag{2}$$

again compatible with filtration, Galois action, and Frobenius. However, most of the paper deals with the following mod-p^n version, still with an almost isomorphism:

$$\alpha : H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} A_{\text{cris}} \simeq H^*_{\text{cris}}(X_n/W_n(k)) \otimes_{W(k)} A_{\text{cris}}, \tag{3}$$

*Notes for a talk in Berkeley's Student Arithmetic Geometry Seminar, on Wiesława Nizioł's paper of the same title.

where the subscript n denotes reduction modulo p^n. Here we assume that $\dim(X/V) < p-2$, and V is absolutely unramified. These hypotheses are necessary, because the derived-mod-p^n versions of the two sides of the equation pick up p^n-torsion from H^{*+1}, and without some assumptions it may be that H^{*+1}_{cris} has more torsion than $H^{*+1}_{\text{ét}}$.

In this talk we will consider the mod-p^n situation, but we will attempt to gloss over some of the technical clutter about torsion.

The ring B_{cris} is a localization of A_{cris}; let's recall the construction of the latter. Begin with the completed algebraic closure C of K, with ring of integers $\mathcal{O}_C$. This has tilt $\mathcal{O}_C^\flat := \lim_{\leftarrow \varphi} \mathcal{O}_C/p$, a perfect $\mathbb{F}_p$-algebra. The period ring A_{inf} is defined to be $W(\mathcal{O}_C^\flat)$. This is equipped with a canonical surjection $\theta : A_{\text{inf}} \to \mathcal{O}_C$, and A_{cris} is defined to be the p-adically completed PD-envelope of the ideal $\ker\theta \subset A_{\text{inf}}$; namely,

$$A_{\text{cris}} = p - \text{adic completion of } \left(A_{\text{inf}} \left[\frac{\alpha^n}{n!} : \alpha \in \ker\theta, n \geq 1 \right] \subset A_{\text{inf}}[1/p] \right) \tag{4}$$

Note in particular that A_{cris} is much larger than $\overline{K}$. This says that some transcendental "periods" are needed to define an isomorphism between étale and crystalline cohomology; these periods have to do with doing calculus on some formal PD-thickening of X_k.

Remark: $A_{\text{cris}}[0]$ can be identified with the crystalline cohomology ring $H^*_{\text{cris}}(\overline{V}/W(k))$, so the crystalline side of the equation can be rewritten as

$$\begin{aligned} H^*_{\text{cris}}(X_k/W(k)) \otimes_{W(k)} A_{\text{cris}} &\cong H^*_{\text{cris}}(X_k/W(k)) \otimes_{W(k)} H^*_{\text{cris}}(\overline{V}_n/W(k)) && (5) \\ &\cong H^*_{\text{cris}}(X_{\overline{V}_n}/W(k)), && (6) \end{aligned}$$

using the Künneth formula. From now on, we will always interpret the crystalline side of the equation like so. So we really want a map from $H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}_p} A_{\text{cris}}$ to $H^*_{\text{cris}}(X_{\overline{V}_n}/W(k))$ inducing an almost isomorphism

$$H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}_p} A_{\text{cris}} \simeq H^*_{\text{cris}}(X_{\overline{V}_n}/W(k)). \tag{7}$$

1.2 Motivation for use of K-theory

At this point, one may ask: if we want a map from étale cohomology to crystalline cohomology, why would we pass through algebraic K-theory? One motivation for this is as follows: for X smooth over a field, there exists a fourth-quadrant spectral sequence

$$E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \implies K_{-p-q}(X) \tag{8}$$

relating Voevodsky's motivic cohomology groups $H^p(X, \mathbb{Z}(-q/2))$ to the K-groups of X.[1] This degenerates after tensoring with $\mathbb{Q}$; in fact, we will impose sufficiently restrictive hypotheses for it to degenerate integrally, which is possible by the work of Soulé. In particular, taking

[1]The coefficient group $\mathbb{Z}(-q/2)$ is by definition some complex of abelian sheaves on X_{Zar}, which is 0 for q odd and $\mathbb{Z}[0]$ for $q = 0$.

$p = j$ and $q = -2i$, we get that $H^j(X, \mathbb{Z}(i))$ is (possibly up to torsion) a piece in a canonical filtration of K_{2i-j}. Since motivic cohomology maps to étale and crystalline cohomology, we expect a diagram of the form

$$\begin{tikzcd} & \text{(filtered piece of K-group)} \arrow[dl, "\bar{c}_{ij}^{\text{cris}}"'] \arrow[dr, "\bar{c}_{ij}^{\text{ét}}"] & \\ \text{(crystalline cohomology)} & & \text{(étale cohomology)} \arrow[ll, dotted, "\alpha"'] \end{tikzcd}$$

(We will be more precise about this later.) It will turn out that $\bar{c}_{ij}^{\text{ét}}$ is surjective and its kernel can be controlled. We then construct the dotted morphism by taking preimages in K-theory, mapping down to crystalline cohomology, and showing that the "error terms" in K-theory map to 0.

This gives us our map. From this point, the rest of the proof is more or less formal: it suffices to check that the map is compatible with all relevant structures as well as Poincaré duality and some cycle classes.

2 K-theory and its comparison to étale cohomology

2.1 Introduction to algebraic K-theory

For X a scheme, the *0th algebraic K-group* $K_0(X)$ is the Grothendieck group of finite-rank vector bundles on X. This was preceded by (and is motivated by) its obvious analogue for topological spaces, with topological $\mathbb{R}$- or $\mathbb{C}$-vector bundles.

Example: If A is a Dedekind domain, $K_0(\operatorname{Spec} A) \cong \mathbb{Z} \oplus \operatorname{Pic} A$, with the $\mathbb{Z}$ indicating rank.

Example: $K_0(\mathbb{P}^n_K) \cong \mathbb{Z}^{\oplus n+1}$. One way to think of this is that K_0 is built out of the motivic cohomology groups $H^{2i}(X, \mathbb{Z}(i)) = CH^i(X)$, which are $\mathbb{Z}$ for $0 \leq i \leq n$ and 0 otherwise. So $K_0(\mathbb{P}^n_K)$ admits a descending filtration F^i_γ whose quotients are $n+1$ copies of $\mathbb{Z}$. Each $F^i_\gamma K_0(\mathbb{P}^n_K)$ can be identified with the Grothendieck group of coherent sheaves on $\mathbb{P}^n_K$ supported in codimension at least i. (Here we are using the fact that the Grothendieck group of vector bundles agrees with that of coherent sheaves, which is true in good situations but not in general.)

Historically, it took a lot of work to find the correct definitions of higher K-theory for a ring, and more work for a scheme. This was finally accomplished by Quillen's Q-construction in the 1970's. For X a scheme, consider the category $\operatorname{Vect}(X)$ of vector bundles on X, a full subcategory of the abelian category of coherent sheaves. The Quillen K-groups of X are $K_i(X) = \pi_i(\Omega BQ\operatorname{Vect}(X))$, the ith homotopy group of the loopspace of the classifying space of the geometric realization of the nerve of $\operatorname{Vect}(X)$.

K-theory forms a ring spectrum, so it is a contravariant functor in the scheme X, behaving like cohomology with coefficients in a ring. One can moreover consider K-theory with coefficients;

this is related to regular K-theory by a universal coefficient theorem (i.e. derived tensor product).

Remark: K-groups are constructed as homotopy groups, and as such they are difficult to calculate, particularly if one is interested in torsion. For example, it is known that K_0 through K_4 of $\operatorname{Spec}\mathbb{Z}$, with $\mathbb{Z}$ coefficients, are $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48$, and 0. But $K_5(\mathbb{Z})$ is only known to be isomorphic to $\mathbb{Z} \oplus$ (some finite 3-torsion group).

2.2 Thomason's comparison between algebraic and étale K-theory

As mentioned before, there is a spectral sequence

$$E_2^{pq} = H^p_{\text{mot}}(X, \mathbb{Z}(-q/2)) \implies K_{-p-q}(X) \tag{9}$$

relating algebraic K-theory to motivic cohomology, and similarly with $\mathbb{Z}/\ell^n$ coefficients on each side. There is also a so-called étale K-theory, which is computed by a similar spectral sequence built out of étale cohomology:

$$E_2^{pq} = H^p_{\text{ét}}(X, \mathbb{Z}/\ell^n(-q/2)) \implies K^{\text{ét}}_{-p-q}(X; \mathbb{Z}/\ell^n), \tag{10}$$

where we are using ℓ instead of p to avoid confusion with the coordinate p in the spectral sequence. There is a canonical map $K_{-p-q}(X; \mathbb{Z}/\ell^n) \to K^{\text{ét}}_{-p-q}(X, \mathbb{Z}/\ell^n)$, and a corresponding map in cohomology $H^p_{\text{mot}}(X, \mathbb{Z}/\ell^n(-q/2)) \to H^p_{\text{ét}}(X, \mathbb{Z}/\ell^n(-q/2))$. We view these as maps of (bi)graded rings:

$$K_*(X; \mathbb{Z}/\ell^n) \to K^{\text{ét}}_*(X; \mathbb{Z}/\ell^n), \tag{11}$$
$$H^*_{\text{mot}}(X, \mathbb{Z}/\ell^n(*')) \to H^*_{\text{ét}}(X, \mathbb{Z}/\ell^n(*')) \tag{12}$$

The big theorem of Thomason (and that of Levine) says that under reasonable hypotheses on X, these maps are not far from being isomorphisms. That is, we can measure how much étale K-theory differs from algebraic K-theory, and similarly how much étale cohomology differs from motivic cohomology.

More precisely: suppose X is smooth and of finite type over a field K that contains a primitive nth root of unity ζ. Then $H^0_{\text{mot}}(\operatorname{Spec} K, \mathbb{Z}/\ell^n(1)) = \mu_{\ell^n}(K)$. This contains the element ζ, which we pull back to $H^0_{\text{mot}}(X, \mathbb{Z}/\ell^n(1))$ to get the so-called *Bott element* β_n. There exists a corresponding element $\beta_n \in K_2(X; \mathbb{Z}/\ell^n)$, roughly because this receives a contribution from $H^0_{\text{mot}}(X, \mathbb{Z}/\ell^n(1))$ in the spectral sequence.

Theorem 2.1. *(Thomason, Levine)*[2] *Suppose X is finite type over a separably closed field of characteristic not p, and $p \neq 2$.*[3] *Then the ring maps (11) and (12) are given by inverting the elements β_n.*

[2] Thomason proved it for K-theory in a 116-page paper, and Levine used Thomason's methods to prove the corresponding result for cohomology.

[3] If $p = 2$, we must also assume that there is a square root of -1 on X.

3 Outline of the construction

We are now ready to tackle Nizioł's construction of the map

$$\alpha : H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n) \to H^*_{\text{cris}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}). \tag{13}$$

In fact we will actually construct it after applying a sufficiently high Tate twist to the coefficients on both sides. (Since crystalline cohomology has no Galois action, the Tate twist on it refers instead to twisting the Hodge filtration and the Frobenius.) First we must impose some hypotheses to get around issues of torsion. Let $d = \dim(X/V)$, let $i > 0$ and $j \geq \frac{4d(d+1)(d+2)}{3}$ be integers, and let the prime p be sufficiently large, possibly depending on d, i, and j.[4]

Consider have the following diagram:

$$\begin{tikzcd}
F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{V}}; \mathbb{Z}/p^n) \arrow[r, "\sim", "j^*"'] \arrow[d, "\overline{c}^{\text{cr}}_{ij}"] & F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{K}}; \mathbb{Z}/p^n) \arrow[d, "\overline{c}^{\text{ét}}_{ij}"] \\
H^{2i-j}_{\text{cris}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}(i)) & H^{2i-j}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n(i)) \arrow[l, dotted, "\alpha_{2i-j,i}"]
\end{tikzcd}$$

Here, the two objects on the top are filtered pieces of algebraic K-theory, under the γ-filtration defined by the spectral sequences discussed earlier. The top map is induced by $X_{\overline{K}} \hookrightarrow X_{\overline{V}}$ and the functoriality of K_j; Nizioł shows that it is an isomorphism. The vertical maps are Chern class maps, which factor through motivic cohomology

Proposition 3.1. *(Nizioł, based on work of Thomason and Soulé) Under the hypotheses above, the map $\overline{c}^{\acute{e}t}_{ij}$ factors as*

$$F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{K}}; \mathbb{Z}/p^n) \to F^i_\gamma/F^{i+1}_\gamma K^{\acute{e}t}_j(X_{\overline{K}}; \mathbb{Z}/p^n) \xrightarrow{\sim} H^{2i-j}(X_{\overline{K}}, \mathbb{Z}/p^n(i)), \tag{14}$$

where the second map is an isomorphism and the first is surjective with kernel killed by a power of the Bott element $\beta_n \in K_2(X_{\overline{K}}; \mathbb{Z}/p^n)$.

The desired map $\alpha_{2i-j,i}$ is then defined by taking preimages in K-theory and then mapping down to crystalline cohomology. It turns out that the image of β_n in the crystalline cohomology ring is a non-zero-divisor, so the error term disappears and $\alpha_{2i-j,i}$ is well-defined. Choosing suitably large i and j allows us to construct this map in any cohomological degree, provided that p is greater than (approximately) $\frac{2d^3}{3}$.

[4]The exact requirement is as follows: $p \geq \max\{3, i+1, \frac{d+j+3}{2}\}$, and $p^n \geq 5$. If we relax this to merely $p \neq 2$ and $p^n \geq 5$, both parts of the claim may be off by T-torsion for some constant $T = T(d, i, j)$.